C*-Algebras of Expansive Dynamical Systems

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Outline

1. Dynamical systems
2. Building C^*-Algebras From Expansive Dynamical Systems
3. Synchronizing Systems
4. Shift Spaces
5. Results

Motivation

- Smale space C^*-algebras are well understood (Putnam, Putnam–Spielberg, etc)
- K. Thomsen has constructed C^*-algebras associated to expansive dynamical systems, but these are not as well understood in the general case.
- Goal is to extend Smale space techniques in the study of expansive dynamical systems.

Dynamical Systems

A ***dynamical system*** is (X, φ) where X is a compact metric space and $\varphi : X \to X$ is a homeomorphism.

Definitions

- A dynamical system (X, φ) is ***expansive*** if there exists a constant $\varepsilon_X > 0$ such that $d(\varphi^n(x), \varphi^n(y)) \leq \varepsilon_X$ for all $n \in \mathbb{Z}$ implies $x = y$.

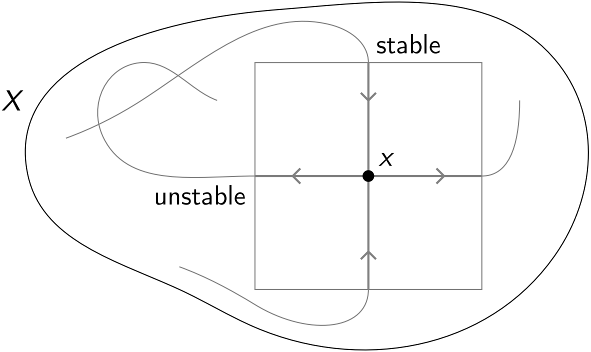
- A dynamical system (X, φ) is ***irreducible*** if for every pair of non-empty open sets U and V there is an $n \in Z$ such that $\varphi^n(U) \cap V \neq \emptyset$.

- Two points $x, y \in X$ are called ***(un)stably*** equivalent if

$$\lim_{n\to\infty} d(\varphi^n(x), \varphi^n(y)) = 0 \text{ (stable)}$$
$$\lim_{n\to\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0 \text{ (unstable)}$$

Smale Spaces

A ***Smale Space*** (X, φ) is a dynamical system that is locally hyperbolic in the sense that each point has a neighborhood which is homeomorphic to the product of its local stable and unstable sets.



Hyperbolic Toral Automorphisms

- Hyperbolic toral automorphisms are a type of Smale space.
- For example, consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

- A is invertible over $\mathbb{Z}$ and induces a homeomorphism

$$\varphi : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

- A has two eigenvalues λ_s, λ_u such that $0 < \lambda_s < 1 < \lambda_u$.
- The eigenvectors corresponding to these eigenvalues determine the stable and unstable directions at a given point.

C^*-Algebras From Expansive Dynamical Systems

Following Thomsen, $x, y \in X$ are called ***locally conjugate*** if there exist open neighborhoods U and V of x and y respectively, and a homeomorphism $\gamma : U \to V$ such that $\gamma(x) = y$ and

$$\lim_{n \to \pm\infty} \sup_{z \in U} d(\varphi^n(z), \varphi^n(\gamma(z))) = 0\,.$$

This is an equivalence relation! Denote by $R \subseteq X \times X$, however we topologize R with subbase

$$\{(z, \gamma(z)) \mid z \in U\}$$

for every U, V, and γ as above.

With this topology R is an étale groupoid, we construct the groupoid C^*-algebra

$$A = C_r^*(R)$$

called the ***homoclinic algebra*** of (X, φ).

Remark: The ***heteroclinic algebras*** S and U are related C^*-algebras also constructed from an expansive dynamical system. For Smale spaces these are the same as the *stable* and *unstable* algebras constructed by Putnam. Note that the construction of these algebras requires periodic points to be dense!

Synchronizing Systems

Motivation: Describe a class of expansive systems with enough points that have local hyperbolic neighborhoods. Some results for Smale spaces only depend on local product structure.

Definition

Local *stable* and *unstable* sets of $x \in X$ are defined as follows.

$$X^{\mathrm{s}}(x, \varepsilon) = \{y \in X \mid d\left(\varphi^{n}(x), \varphi^{n}(y)\right) \leq \varepsilon \text{ for all } n \geq 0\}$$
$$X^{\mathrm{u}}(x, \varepsilon) = \left\{y \in X \mid d\left(\varphi^{-n}(x), \varphi^{-n}(y)\right) \leq \varepsilon \text{ for all } n \geq 0\right\}$$

For $0 < \varepsilon \leq \frac{\varepsilon_X}{2}$ the intersection $X^{\mathrm{s}}(x, \varepsilon) \cap X^{\mathrm{u}}(y, \varepsilon)$ consists of at most one point (by expansiveness!).

Synchronizing Systems

For $0 < \varepsilon \leq \frac{\varepsilon_X}{2}$ define:

$$D_\varepsilon = \{(x, y) \in X \times X \mid X^{\mathrm{s}}(x, \varepsilon) \cap X^{\mathrm{u}}(y, \varepsilon) \neq \emptyset\}$$

and a map $[-, -] : D_\varepsilon \to X$ such that $[x, y] \in X^{\mathrm{s}}(x, \varepsilon) \cap X^{\mathrm{u}}(y, \varepsilon)$.

Notes:

- $[-, -]$ is continuous
- D_ε is closed and contains $\Delta_X = \{(x, x) \mid x \in X\}$.

Definition

A point $x \in X$ is called ***synchronizing*** if there exists $\delta_x > 0$ such that

$$X^{\mathrm{u}}(x, \delta_x) \times X^{\mathrm{s}}(x, \delta_x) \subseteq D_\varepsilon$$

and $[-, -]$ restricted to $X^{\mathrm{u}}(x, \delta_x) \times X^{\mathrm{s}}(x, \delta_x)$ is a homeomorphism onto its image, which is a neighborhood of x.

Synchronizing Systems

An expansive dynamical system (X, φ) is called a ***sychronizing system*** if it is *irreducible* and there exists a synchronizing point $x \in X$.

Remarks:

- Smale space are synchronizing systems where every point is synchronizing.
- By irreducibility, synchronizing systems have a dense open set of synchronizing points.
- There exist expansive dynamical systems that are *not* synchronizing, e.g. *minimal* (every orbit is dense) expansive dynamical systems such as Toeplitz flows.
- *Synchronizing shifts* have been studied in symbolic dynamics.

Shift Spaces

Let $\mathcal{A}$ be a finite set, consider the space $\mathcal{A}^{\mathbb{Z}} = \{(x_i)_{i\in\mathbb{Z}} \mid x_i \in \mathcal{A}\}$. The *shift map* $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by

$$\sigma(x)_i = x_{i+1}$$

Definitions

- A ***shift space*** is a closed subspace $X \subseteq \mathcal{A}^{\mathbb{Z}}$ which is invariant under σ.
- For a shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$, the set of finite words appearing in any element of X is denoted $\mathcal{L}(X) \subseteq \bigcup_{n\geq 0} \mathcal{A}^n$ and is called the *language* of X.

We think of (X, σ) as a dynamical system. Shift spaces are expansive!

Shift Spaces

Examples:

- Shift spaces that are also Smale spaces are called *shifts of finite type*. A shift of finite type can be constructed as the set of sequences in $\mathcal{A}^{\mathbb{Z}}$ which do not contain any of the words in a finite set of forbidden words denoted $\mathcal{F} \subseteq \bigcup_{n \geq 0} \mathcal{A}^n$.
    - The full 2-shift is $\{0, 1\}^{\mathbb{Z}}$.
    - The "golden mean shift" is the set of all sequences in $\{0, 1\}^{\mathbb{Z}}$ which do not contain consecutive 1's.
- Shift spaces that are also synchronizing are called *synchronizing shifts*. These have been studied in symbolic dynamics.
    - The even shift is the set of all sequences in $\{0, 1\}^{\mathbb{Z}}$ which have an even number of consecutive 0's between any 1's.

Topology on Shift Spaces

- In the topology on a shift space, two points x and y are close together if

$$x_{[-N,N]} = x_{-N}x_{-N+1}\cdots x_{N-1}x_N = y_{[-N,N]}$$

- Two points $x, y \in X$ are (un)stably equivalent in a shift space if for some $N \in \mathbb{Z}$

$$x_n = y_n \text{ for all } n \geq N \text{ (stable)}$$
$$x_n = y_n \text{ for all } n \leq N \text{ (unstable)}$$

- The local stable and unstable sets are

$$X^s(x,\varepsilon) = \{y \in X \mid y_n = x_n \text{ for all } n \geq N_\varepsilon\}$$
$$X^u(x,\varepsilon) = \{y \in X \mid y_n = x_n \text{ for all } n \leq N_\varepsilon\}$$

Shift Spaces

- Let $x \in X$ be synchronizing, $\delta_x > 0$ such that for $y \in X^u(x, \delta_x), z \in X^s(x, \delta_x)$ we have the following:

$$\begin{aligned} y &= (\dots x_{-N-2}x_{-N-1})(x_{-N}x_{-N+1}\dots x_{N-1}x_N)(y_{N+1}y_{N+2}\dots) \\ z &= (\dots z_{-N-2}z_{-N-1})(x_{-N}x_{-N+1}\dots x_{N-1}x_N)(x_{N+1}x_{N+2}\dots) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\downarrow \\ [y,z] &= (\dots z_{-N-2}z_{-N-1})(x_{-N}x_{-N+1}\dots x_{N-1}x_N)(y_{N+1}y_{N+2}\dots) \end{aligned}$$

- In other words, x is synchronizing if there is an N large enough that anything that can happen on the left and right of $x_{[-N,N]}$ can be "glued together".
- A synchronizing word in a shift space X is a word w such that if u, v are words such that $uw, wv \in \mathcal{L}(X)$, then $uwv \in \mathcal{L}(X)$. Synchronizing shifts are irreducible shift spaces which contain a synchronizing word.

Shift Spaces

Local conjugacy relation for shift spaces:

- Two points $x, y \in X$ are locally conjugate if for some N large enough, there is a bijection constructed as follows. Let z satisfy $z_{[-N,N]} = x_{[-N,N]}$.

$$z = (\dots z_{-N-2} z_{-N-1})(x_{-N} x_{-N+1} \dots x_{N-1} x_N)(z_{N+1} z_{N+2} \dots)$$
$$\downarrow \gamma$$
$$\gamma(z) = (\dots z_{-N-2} z_{-N-1})(y_{-N} y_{-N+1} \dots y_{N-1} y_N)(z_{N+1} z_{N+2} \dots)$$

- We can construct the homoclinic algebra of a shift space by considering equivalence classes of words arising from the bijection above.

Even Shift

Let $X \subseteq \{0,1\}^{\mathbb{Z}}$ be the set of all elements of $\{0,1\}^{\mathbb{Z}}$ which do not contain the word $10^{2k+1}1$ for any $k \geq 0$. This is a shift space called the ***even shift***.

The even shift is a *sofic shift* (not a Smale space!).

Consider the sequence of all zeros:

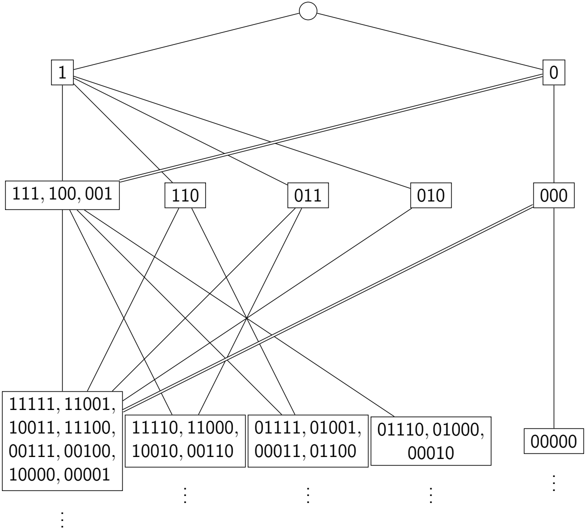
$$\overline{0} = \ldots 00000 \ldots \in X$$

This point is *not* synchronizing! Let $x \in X^{\mathrm{u}}(\overline{0}, \varepsilon)$ and $y \in X^{\mathrm{s}}(\overline{0}, \varepsilon)$.

$$\begin{aligned} x &= \ldots 001000000 \ldots \\ y &= \ldots 000000100 \ldots \\ [x, y] &= \ldots 001000100 \ldots \end{aligned}$$

Even Shift

Bratteli diagram for the homoclinic algebra of the even shift



Even Shift

- The Bratteli diagram determines the homoclinic algebra A of the even shift.
- The K-theory of A can be computed as

$$K_0(A) = \varinjlim \left\{ \mathbb{Z}^5 \xrightarrow{P} \mathbb{Z}^5 \xrightarrow{P} \dots \right\}$$

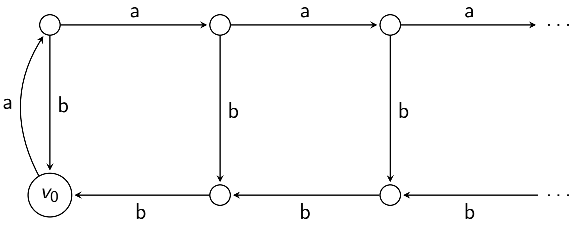
where

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is the matrix encoding the edge relations in the Bratteli diagram.

$a^n b^n$-Shift

Let $X \subseteq \{a, b\}^{\mathbb{Z}}$ be the closure of the set of bi-infinite paths on the following graph



This is a synchronizing shift.

A word in $\mathcal{L}(X)$ is synchronizing if it is represented by a path that on the graph passing through v_0.

Results

For a dynamical system (X, φ) the set of periodic points is defined as

$$\mathrm{Per}(X, \varphi) = \{x \in X \mid \varphi^n(x) = x \text{ for some } n > 0\}$$

Theorem (Deeley, S.)

If (X, φ) is a synchronizing system then $\mathrm{Per}(X, \varphi)$ is dense in X.

Theorem (Deeley, S.)

The homoclinic algebra of an expansive dynamical system is asymptotically commutative. That is, for each $a, b \in A$

$$\lim_{n \to \infty} ||\varphi^n(a)b - b\varphi^n(a)|| = 0\,.$$

Results

We can think of the K-theory of the C^*-algebras A, S, and U as giving information about what type of expansive system (X, φ) is.

Theorem

For an expansive dynamical system (X, φ), if $S \otimes U$ is not Mortia equivalent to A then (X, φ) is not a Smale space.

- For example, this is not true for the even shift, so it cannot be a shift of finite type.

Theorem

For a shift space (X, σ), if the rank of $K_0(A)$ is not finite then X cannot be a sofic shift.

- The $a^n b^n$-shift has infinite rank K-theory, and is not a sofic shift.

Results

Theorem

If (X, φ) is a synchronizing system and $x, y \in X$ are locally conjugate, then x is synchronizing if and only if y is synchronizing. Hence we have an ideal $\mathcal{I}_{\text{sync}} \subseteq A$ and a short exact sequence

$$0 \longrightarrow \mathcal{I}_{\text{sync}} \longrightarrow A \longrightarrow A/\mathcal{I}_{\text{sync}} \longrightarrow 0$$

This ideal has similar properties to a Smale space C^*-algebra, for example we conjecture that $\mathcal{I}_{\text{sync}}$ is Morita equivalent to $S \otimes U$ where S and U are the stable and unstable algebras of Smale spaces.

- For example the even shift has only one non-synchronizing point, so $A/\mathcal{I}_{\text{sync}} \cong \mathbb{C}$.
- Expansive homeomorphisms on surfaces have only a finite number of non-synchronizing points, and so $A/\mathcal{I}_{\text{sync}}$ is a finite dimensional C^*-algebra.

Results

For the even shift, the stable algebra S and the unstable algebra U can be computed via Bratteli diagrams similar to the previous one, where

$$P^s = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad P^u = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The ideal $\mathcal{I}_{\text{sync}}$ in the even shift is built from the vertices in the Bratteli diagram representing equivalence classes of synchronizing words. In particular we have the following.

$$P_{\text{sync}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = P^s_{\text{sync}} \otimes P^u_{\text{sync}}$$

References

- K. Thomsen, *C^*-Algebras of Homoclinic and Heteroclinic Structure in Expansive Dynamical Systems*
- D. Fried, *Finitely Presented Dynamical Systems*
- I. Putnam, *C^*-Algebras From Smale Spaces*
- D. Ruelle, *Thermodynamic Formalism*

Thank you!