Index Theory for Groupoid-Equivariant Operators

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Introduction

This talk pertains to unfinished work for my PhD thesis at Dartmouth College under Erik van Erp.

Theorems that are not published or fully proven are labeled as "Work in progress".

Please feel free to interrupt and ask questions throughout.

Overview

What is an Index Theorem?

For a **Fredholm operator** F, there is a well-defined **index**:

$$ind(F) = dim(ker(F)) - dim(coker(F)).$$

This index is **highly stable**:

1. If F_t is a homotopy of Fredholm operators, then $ind(F_t)$ is constant.
2. If K is a compact operator, then $F + K$ is also Fredholm, and $ind(F + K) = ind(F)$.

An **index theorem** is a topological description of the index, for a specific class of Fredholm operators (e.g. elliptic ΨDO).

Atiyah-Singer Index Theorem

Let $P : \Gamma^\infty(E^0) \to \Gamma^\infty(E^1)$ be an **elliptic pseudodifferential operator**, where $E^j \to X$ are smooth, complex vector bundles on a closed, smooth manifold, X.

1. P is (unbounded) Fredholm.
2. Given a measure on X and metric on E^i, an **order zero** ΨDO extends uniquely to a bounded operator $L^2(E^0) \to L^2(E^1)$.
3. P has a **principal symbol**: $\sigma_P : T^*X \to \pi^* Hom(E^0, E^1)$ with $\sigma_P(\xi)$ invertible for $\xi \neq 0$. Therefore, $[\sigma_P] \in K^0(T^*X)$.

Theorem (Atiyah-Singer Formula)

For the operator P defined above,

$$ind(P) = [ch([\sigma_P]) \wedge Td^c(TX)](TX)$$

Kasparov's KK-theory

A and B: $\mathbb{Z}/2\mathbb{Z}$-graded C^*-algebras (separable).

(E, ϕ, F) is an A-B-**Fredholm Module** if:

1. E is a $\mathbb{Z}/2\mathbb{Z}$-graded right Hilbert B-module
2. $\phi : A \to \mathcal{L}(E)$ is a (degree 0) $*$-hom.
3. $F \in \mathcal{L}(E)$ (degree 1) such that, $\forall a \in A$:
   1. $(F^2 - 1)\phi(a) \in \mathcal{K}(E)$
   2. $(F - F^*)\phi(a) \in \mathcal{K}(E)$
   3. $[F, \phi(a)] \in \mathcal{K}(E)$.

$E(A, B) :=$ unitary equivalence classes of Fredholm Modules.
$KK(A, B) := E(A, B)/$"homotopy".

KK-theory Special Cases

Special Cases:

1. *Topological K-theory:* $K^0(X) \cong KK(\mathbb{C}, C_0(X))$,
2. *C^*-algebra K-theory:* $K_0(A) \cong KK(\mathbb{C}, A)$,
3. *K-homology:* $K^0(A) \cong KK(A, \mathbb{C})$,
4. *geometric K-homology:* $k_0(X) \cong KK(C_0(X), \mathbb{C})$,
5. $KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$.

$$\left(E_0 \oplus E_1, \phi, \begin{pmatrix} 0 & F \\ S & 0 \end{pmatrix}\right) \in KK(\mathbb{C}, \mathbb{C}) \leftrightarrow ind(F) \in \mathbb{Z}.$$

Throughout: $X =$LCH space and $A =$ graded (separable) C^*-algebra.

KK-product

There is a product in KK-theory:

$$\underset{B}{\widehat{\otimes}} : KK(A, B) \otimes KK(B, C) \to KK(A, C)$$

1. The KK-product is associative
2. It generalizes K-theory products
3. It generalizes K-homology products
4. Very roughly,

$$(E, \phi_A, F)\underset{B}{\widehat{\otimes}}(E', \phi_B, F') = (E \underset{\phi_B}{\widehat{\otimes}} E', \phi_A \widehat{\otimes} 1, F'')$$

Where F'' is a way of formalizing the (technically nonsense) expression "$F\widehat{\otimes}1 + 1\widehat{\otimes}F'$".

Index Theory in K-Homology

1. An elliptic ΨDO, P, forms a class $[P] \in KK(C(X), \mathbb{C})$
2. σ_P forms a K-theory class $[(\sigma_P, \pi^* E^0, \pi^* E^1)] \in K^0(T^*X)$.
3. Let $[\sigma_P]$ be the K-theory class viewed as an element of $KK(C(X), C_0(T^*X))$.
4. T^*X is almost complex. If D is a Dirac (e.g. Dolbeault) operator on T^*X, then form $[D] \in KK(C_0(T^*X), \mathbb{C})$.

Theorem (Reduction to Dirac)

If P is an elliptic ΨDO on a compact manifold X, then in $KK(C(X), \mathbb{C})$,

$$[P] = [\sigma_P] \underset{C_0(T^*X)}{\widehat{\otimes}} [D]$$

Index Theory in K-Homology (Continued)

The formula:

$$[P] = [\sigma_P] \underset{C_0(TX)}{\widehat{\otimes}} [D]$$

can be translated to the Atiyah-Singer formula:

$$ind(P) = [ch(\sigma_P) \wedge Td^c(TX)](TX)$$

By using the Chern character, Poincare duality, and an embedding of X into $\mathbb{R}^n$.

Baum-Douglas Δ

An alternative formulation of the Atiyah-Singer index theorem is as a commutative diagram:

$$\begin{array}{ccc} & K^0(TX) & \\ {}^{c}\swarrow & & \searrow^{Op} \\ k_0(X) & \xrightarrow[\mu]{} & K_0(C(X), \mathbb{C}) \end{array}$$

$k_0(X)$ is formed out of equivalence classes of triples: (M, E, φ):

1. M closed spin^c-manifold
2. $E \to M$ a $\mathbb{C}$-v.b.
3. $\varphi : M \to X$ continuous.

Baum-Douglas Δ

$$\begin{array}{ccc} & K^0(TX) & \\ {}^{c}\swarrow & & \searrow^{Op} \\ k_0(X) & \xrightarrow[\mu]{\qquad\qquad} & K_0(C(X),\mathbb{C}) \end{array}$$

1. "Clutching":

$$c(\sigma, E, F) := \left(\Sigma X, E \bigcup_{\sigma} F, \pi_{\Sigma X} \right)$$

,

2. "twisting":

$$\mu(M, E, \phi) := [E]_{\phi} \underset{C(M)}{\widehat{\otimes}} [D_M]$$

3. "Choose-an-operator" satisfies:

$$Op(\sigma_P) = [P]$$

Baum-Douglas Δ

$$\begin{array}{ccc} & K^0(TX) & \\ {\scriptstyle c}\swarrow & & \searrow{\scriptstyle Op} \\ k_0(X) & \xrightarrow[\mu]{} & K_0(C(X),\mathbb{C}) \end{array}$$

Let $(\sigma_P, \pi^*E^0, \pi^*E^1) \in K^0(TX)$ be the symbol of an elliptic ΨDO, then the commutativitiy $Op([\sigma_P]) = (\mu \circ c)([\sigma_P])$ reads:

$$[P] = \left[\pi^*E^0 \bigcup_{\sigma_P} \pi^*E^1\right]_{\pi} \underset{\Sigma X}{\widehat{\otimes}} [D_{\Sigma X}]$$

This is essentially the same formula as $[P] = [\sigma_P] \underset{TX}{\widehat{\otimes}} [D_{TX}]$, but treated on a compactification of TX instead of TX itself.

Groupoids

We want to prove similar types of index theorems in the setting where a groupoid is acting on everything, and the elliptic ΨDO is equivariant.

To prep, we will briefly discuss the following ideas:

1. $\mathcal{G}$-manifold
2. $\mathcal{G}$-equivariant elliptic ΨDO on a $\mathcal{G}$-manifold
3. $\mathcal{G} - C^*$-algebra
4. $\mathcal{G}$-equivariant KK-theory (Le-Gall)
5. $\mathcal{G}$-equivariant representable K-theory (Emerson & Meyer)

$\mathcal{G}$-manifolds

$\mathcal{G}$ = proper, LCH topological groupoid,
$Z := \mathcal{G}^{(0)}$,
X= proper, LCH $\mathcal{G}$-space, with anchor map $\rho : X \to Z$.

Definition

X is a $\mathcal{G}$-**manifold** if

1. The fibers of $\rho : X \to Z$ are smooth manifolds, determined by charts of the form $\varphi : U \to \rho(U) \times \mathbb{R}^n$, for open $U \subseteq X$.
2. All change of coordinate maps are continuous
3. all ρ-fiberwise derivatives of change of coordinates should exist and be continuous.
4. $\mathcal{G}$ acts via diffeomorphisms.

Equivariant ΨDO

Very roughly, $P : \Gamma^\infty(E^0) \to \Gamma^\infty(E^1)$ is a Pseudodifferential operator between smooth $\mathcal{G}$-bundles E^0, E^1 over $\mathcal{G}$-manifold X if

1. P fibers as a family of pseudodifferential operators $\{P_z\}_{z \in Z}$ on the manifolds $\rho^{-1}(z) = X_z$.
2. The (full) symbols of P_z, in a restricted coordinate chart $(U \cap \rho^{-1}(z), \psi|_z)$ for X, come from a single (full) symbol for P, in (U, ϕ), that is $C^{\infty,0}$ (suitably defined).

It is proven (by A. Paterson) that these ΨDO form a $KK^{\mathcal{G}}$-class when $\mathcal{G}$ is a proper, continuous family groupoid acting on a proper, cocompact $\mathcal{G}$-manifold.

$C_0(Z)$-Algebras

Definition

A $C_0(Z)$-**algebra** is a C^*-algebra A with a map $\theta : C_0(Z) \to \mathcal{Z}(\mathcal{M}(A))$ such that $\theta(C_0(Z)).A = A$.
The fiber above $z \in Z$ is defined to be $A_z := A/\theta(I_z).A$, where $I_z \subseteq C_0(Z)$ is the ideal of functions vanishing at $z \in Z$.

1. Example: If $f : X \to Y$ is a continuous map, then $C_0(X)$ is a $C_0(Y)$-algebra (via $\theta = f^* : C_0(Y) \to C_b(X)$).
2. If $f : X \to Y$ is a continuous map and A is a $C_0(Y)$-algebra, then define $f^*A := A \widehat{\underset{C_0(Y)}{\otimes}} C_0(X) = A \otimes_{max} C_0(X)|_{X \times_{id,f} Y}$

$\mathcal{G}$-C^*-algebras

Let $\mathcal{G}$ be a LCH groupoid with Haar system, and let A be a $C_0(Z)$-algebra ($Z = \mathcal{G}^{(0)}$).

Definition

A **continuous action of** $\mathcal{G}$ **on** A is a $C_0(\mathcal{G})$-algebra map $\alpha : s^*A \to r^*A$ (restricting to $\alpha_\gamma : A_{s(\gamma)} \to A_{r(\gamma)}$ for all $\gamma \in \mathcal{G}$). such that, $\forall(\gamma, \gamma') \in \mathcal{G}^{(2)}$, the diagram commutes:

$$\begin{array}{ccccc} & & A_{r(\gamma')=s(\gamma)} & & \\ & \overset{\alpha_{\gamma'}}{\nearrow} & & \overset{\alpha_\gamma}{\searrow} & \\ A_{s(\gamma')} & & \xrightarrow[\alpha_{\gamma\circ\gamma'}]{} & & A_{r(\gamma)} \end{array}$$

The pair (A, α) is called a $\mathcal{G}$-C^*-**algebra.**

Le-Gall's $KK^{\mathcal{G}}$

Definition (Le-Gall)

For $\mathcal{G}$-C^*-algebras A and B, define $\mathcal{G}$-**Fredholm Modules**, $E_{\mathcal{G}}(A, B)$, to be elements $(E, \phi, F) \in E(A, B)$ such that E has a $\mathcal{G}$-action (represented by the unitary $V : s^*E \to r^*E$) satisfying:

$$\forall a \in r^*A,\ \phi(a)(V(s^*F)V^* - r^*F) \in r^*\mathcal{K}(E)$$

Define $KK^{\mathcal{G}}(A, B) := E_{\mathcal{G}}(A, B)/$"homotopy".

Notes: A homotopy is an element of $E_{\mathcal{G}}(A, B[0, 1])$.
The extra condition roughly says that the operator F is equivariant modulo $\mathcal{K}(E)$.

Emerson & Meyer's $RK_{\mathcal{G},Y}(X)$

Definition

If $f : X \to Y$ is a continuous map between LCH spaces, then $A \subseteq X$ **is** Y**-compact** if $f\,|_A$ is proper.

Definition

The **representable** K**-theory of** X **with** Y**-compact support** is:

$$RK_{\mathcal{G},Y}(X) := KK^{\mathcal{G}\ltimes Y}(C_0(Y), C_0(X)).$$

A $K_{\mathcal{G}}$-theory defined in terms of $\mathcal{G}$-bundles exists ($VK_{\mathcal{G},Y}(X)$), but the natural map $VK_{\mathcal{G},Y}(X) \to RK_{\mathcal{G},Y}(X)$ is not surjective.

Example: If $P : \Gamma^\infty(E^0) \to \Gamma^\infty(E^1)$ is a $\mathcal{G}$-equivariant ΨDO on proper, cocompact $\mathcal{G}$-manfiold X, then σ_P forms a class in $RK_{\mathcal{G},X}(TX)$.

An index formula in $KK^{\mathcal{G}}$

For the remainder of the talk: Let $\mathcal{G}$ be a proper, continuous family groupoid(*) with Haar system, let X be a proper, cocompact $\mathcal{G}$-manifold.

Theorem (Work in Progress)

If $P : \Gamma^\infty(E^0) \to \Gamma^\infty(E^1)$ *is a* $\mathcal{G}$*-equivariant, ellitpic* ΨDO *on* X*, then*

$$[P] = [\sigma_P] \underset{C_0(TX)}{\widehat{\otimes}} [D_{TX}] \quad \in KK^{\mathcal{G}}(C_0(X), C_0(Z)).$$

Here, $[\sigma_P] \in RK^0_{\mathcal{G},X}(TX) := KK^{\mathcal{G}\ltimes X}(C_0(X), C_0(TX))$*, and* $[D_{TX}] \in KK^{\mathcal{G}}(C_0(TX), C_0(Z))$

Simplifying Dirac

To factor the Dirac class, Emerson and Meyer use a special embedding of X to define a topological "shriek" map. This works whenever X is cocompact and $Z = \mathcal{G}^{(0)}$ has "enough $\mathcal{G}$-bundles".

Theorem (Emerson and Meyer)

Let $\rho : X \to Z$ be a proper, cocompact $\mathcal{G}$-manifold, and suppose Z has enough $\mathcal{G}$-bundles. Then, in $KK^{\mathcal{G}}$,

$$[D_{TX}] = (\rho_{TX})! = (\rho \circ \pi)!$$

Theorem (Alt. Index Theorem (Work-in-progress))

If P is a $\mathcal{G}$-equivariant, ellitpic ΨDO on X, then

$$[P] = [\sigma_P] \underset{C_0(TX)}{\widehat{\otimes}} (\rho \circ \pi)! \quad \in KK^{\mathcal{G}}(C_0(X), C_0(Z)).$$

What is the index??

With no $\mathcal{G}$-action, and working with compact X, $[P] \in KK(C(X), \mathbb{C})$ can be mapped to $ind(P) \in KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$.

In the $\mathcal{G}$-equivariant case,

$$KK^{\mathcal{G}}(C_0(X), C_0(Z)) \longrightarrow KK^{\mathcal{G}}(C_0(Z), C_0(Z)) \cong K_0(C^*_{red}(\mathcal{G})).$$

The natural recepticle for $ind(P)$ is $K_0(C^*_{red}(\mathcal{G}))$.

Baum-Douglass Δ

An analouge for the Baum-Douglass triangle in the $\mathcal{G}$-equivariant setting should look like:

$$\begin{array}{ccc} & RK^0_{\mathcal{G},X}(TX) & \\ {\scriptstyle c}\swarrow & & \searrow{\scriptstyle Op} \\ kk^0_{\mathcal{G}}(X,Z) & \xrightarrow[\mu]{\qquad\qquad} & KK^{\mathcal{G}}_0(C_0(X),C_0(Z)) \end{array}$$

1. $kk^0_{\mathcal{G}}(X,Z)$ is Emerson & Meyer's geometric *kk*-theory via correspondences.
2. Emerson & Meyer prove μ is an isomorphism when X is a proper $\mathcal{G}$-manifold, and Z has enough $\mathcal{G}$-bundles.
3. As far as I am aware, the *Op*-map has not been defined yet in the literature. Part of the difficulty: $VK \neq RK$.

Concluding Remarks

1. There are existing index theorems for proper, cocompact Lie groupoid actions (Pflaum, Posthuma, Tang). They define an index pairing between groupoid cohomology classes and equivariant operators.
2. The original purpose of this project was to eventually prove a $\mathcal{G}$-equivariant index theorem for a specific class of hypoelliptic operators on contact manifolds. The non-equivariant case was proved by Baum and van Erp.