Lawson-Pierce Duality between Ample Groupoid Bundles and Steinberg Rings & Semigroups

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University of Colorado, Colorado Springs

Algebraic-Topological Dualities: Rings and Algebras

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Boolean Inverse Semigroups $\leftrightarrow$ Ample Groupoids.

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- Could we even use these to derive their ring/algebra analogs?

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- Then $\mathcal{CO}(X)$ is a Boolean algebra $\because \forall\ O, N, M \in \mathcal{CO}(X)$

$$\emptyset \subseteq O \subseteq X \qquad \text{(Bounded)}$$
$$O \wedge N = O \cap N \qquad \text{(Meets/Infima)}$$
$$O \vee N = O \cup N \qquad \text{(Joins/Suprema)}$$
$$O^c = X \setminus O \qquad \text{(Complements)}$$
$$M \wedge (N \vee O) = (M \wedge N) \vee (M \wedge O) \qquad \text{(Distributive)}$$

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is a maximal proper down-directed up-set (= ultrafilter):

$$\begin{array}{rcll} & \emptyset \notin \mathcal{U}_x & & \text{(proper)} \\ O \supseteq N \in \mathcal{U}_x & \Rightarrow & O \in \mathcal{U}_x & \text{(up-set)} \\ O, N \in \mathcal{U}_x & \Rightarrow & O \cap N \in \mathcal{U}_x & \text{(down-directed)} \\ O \in \mathcal{U}_x & \text{or} & X \setminus O \in \mathcal{U}_x & \text{(maximal)} \end{array}$$

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- Moreover, $O \in \mathcal{CO}(X)$ gets mapped to $\{\mathcal{U}_x : O \in \mathcal{U}_x\}$.
- Topologising ultrafilters like so, $x \mapsto \mathcal{U}_x$ is a homeomorphism.

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- Identifying each $U \in \mathcal{U}(B)$ with $\mathbf{1}_U : B \to \{0,1\}$ where

$$\mathbf{1}_U(a) = \begin{cases} 1 & \text{if } a \in U \\ 0 & \text{if } a \notin U, \end{cases}$$

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Theorem (Stone 1936)

Boolean algebras are dual to Stone spaces via

$$B \mapsto \mathcal{U}(B) \qquad \text{and} \qquad X \mapsto \mathcal{CO}(X).$$

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- We denote the bisections or slices of G by
  $$\mathcal{B}(G) = \{B \subseteq G : \mathsf{r} \text{ and } \mathsf{s} \text{ are injective on } B\}.$$

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- If $O, N \in \mathcal{B}^\circ_\mathrm{c}(G)$ then $O \setminus N, O \cap N \in \mathcal{B}^\circ_\mathrm{c}(G)$. If $O \perp N$ then $O \cup N \in \mathcal{B}^\circ_\mathrm{c}(G)$ too so $\mathcal{B}^\circ_\mathrm{c}(G)$ is a Boolean inverse semigroup.

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Theorem (Lawson 2010)

Boolean inverse semigroups are dual to ample groupoids via
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- Bistable shiftable expectations will soon play a greater role...

Ample Category Bundles

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- If G is an ample groupoid, ρ has a zero section and each fibre has invertibles ($\rho[C^\times] = G$), ρ is an ample category bundle.

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$\therefore$ We need is some additional algebraic structure on $\mathcal{S}_c(\rho)$...

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- For any other $s \in \mathcal{S}_\mathsf{c}(\rho)$, we also immediately see that
$$(a \vee b)s = as \vee bs \quad \& \quad s(a \vee b) = sa \vee sb. \quad \text{(Distributivity)}$$

Orthosuprema

- On well-structured (S, Z, Φ) with $0 \in Z$ we define $\perp$ by

$$a \perp b \quad \Leftrightarrow \quad \exists y, z \in Z \ (ya = a = az \text{ and } yb = 0 = bz).$$

- For inverse semigroups, $a \perp b$ means $ab^{-1} = a^{-1}b = 0$.
- If $\rho : C \twoheadrightarrow G$ is an ample category bundle with well-structured $(S, Z, \Phi) = (\mathcal{S}_c(\rho), \mathcal{Z}_c(\rho), \Phi^\rho_c)$ then, for any $a, b \in \mathcal{S}_c(\rho)$,

$$a \perp b \quad \Leftrightarrow \quad \mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset \ \& \ \mathrm{supp}(a) \cup \mathrm{supp}(b) \in \mathcal{B}(G).$$

$\Rightarrow$ we can define a supremum (w.r.t. restriction) $a \vee b \in \mathcal{S}_c(\rho)$ by

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- Also $y, z \in \mathcal{Z}_c(\rho)$, i.e. $y[\mathrm{supp}(y)], z[\mathrm{supp}(z)] \subseteq C^0$,
  $\Rightarrow \ y \vee z \in \mathcal{Z}_c(\rho)$.

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- Note G here is Hausdorff but C may not be. In fact

$$C \text{ is Hausdorff} \quad \Leftrightarrow \quad \text{each } a \in \mathcal{S}_{\mathsf{c}}(\rho) \text{ has open support}$$
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$\Rightarrow$ (S, Z) is a restriction semigroup (see Kudryavtseva-Lawson).

Boolean Restriction $\wedge$-Semigroups $\subsetneq$ Steinberg Semigroups.

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- Thus the fibre at g can be recovered from the equivalence classes of $\mathcal{S}_\mathsf{c}(\rho)$ modulo the relation defined on the right.

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- This ρ is an ample category bundle with the topology on $\mathcal{U}[S]$ generated by $\rho^{-1}[\mathcal{U}(a)]$ and $\mathcal{U}[a] = \{[a, U] : U \in \mathcal{U}(S)\}$.

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Theorem (B. 2021)

Steinberg semigroups are dual to ample category bundles via

$$(S, Z, \Phi) \mapsto \rho_{(S,Z,\Phi)} \qquad \text{and} \qquad \rho \mapsto (\mathcal{S}_\mathrm{c}(\rho), \mathcal{Z}_\mathrm{c}(\rho), \Phi^\rho_\mathrm{c}).$$

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Theorem (B. 2021)

Under these morphisms, Steinberg semigroups and ample category bundles form equivalent categories.

Ample Ringoid Bundles → Steinberg Rings

Ample Ringoid Bundles $\to$ Steinberg Rings

Definition

An ample ringoid bundle is an ample category bundle $\rho : C \twoheadrightarrow G$ where each fibre $\rho^{-1}\{g\}$ is an abelian group and products distribute over sums whenever they are defined, i.e.

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$\Rightarrow$ $(\mathcal{C}_c(\rho), \mathcal{S}_c(\rho), \mathcal{Z}_c(\rho), \Phi_c^\rho)$ is a Steinberg ring, i.e.
a Steinberg semigroup generating a larger ring.

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Theorem (B. 2021)

Steinberg rings & ample ringoid bundles form equivalent categories.